AMERICAN UNIV. OF BEIRUT

QUIZ-I-MATH.101 **Oct. 29, 2011**

(Material Covering Sec. 1.1------Sec.3.2 of Thomas' Calculus 12^{TH} Ed.)

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NAME:.....SOLUTIONS........ I.D.......................................

SECTION: KHACHADOURIAN 5. 6. 7. 8.

BRIGHT 9. 10. 11.

Question	1	2	3	4	5	6	7	8	TRUE FALSE	QUIZ I GRADE
Max.	6	6	16	6	16	6	12	8	24	100
Gr.										

1. (6 Points) Suppose $f(x)=\begin{cases} x^2 & if\ \ x<2 \\ 3 & if\ \ x=2 \\ 3x-3 & if\ \ x>2 \end{cases}$

   Is this function continuous at $x=2$? Justify your response.

To be continuous, we need $\lim_{x\to 2} f(x)$ to exist and be equal to $f(2)$.

We have: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x^2) = 4$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(3x-3) = 3$

so $\lim_{x\to 2} f(x)$ does not exist, so f is not continuous at 2.

2. (3+3=6 Points) You are given the function : $f(x)=5x+1+\dfrac{7}{x-2}$

   a) Is there a vertical asymptote to the graph of this function? If yes, give its equation.

Yes, $x=2$.

b) Is there an oblique asymptote to the graph? If yes, give its equation.

Yes, $y = 5x + 1$

3. (5+3+3+5=16 Points) Given the function : $y = f(x) = \frac{1}{x^2}; x \neq 0$.

a) Apply the formula: $f'(x) = \lim_{h \to 0}[\frac{f(x+h) - f(x)}{h}]$,

to show that the derivative of the function at every point of its domain is $f'(x) = \frac{-2}{x^3}$.

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^{-2} - x^{-2}}{h}$$

$$= \lim_{h \to 0} \frac{x^2 - (x^2 + 2xh + h^2)}{hx^2(x+h)^2}$$ (putting over a common denominator)

$$= \lim_{h \to 0} \frac{-2x + h}{x^2(x+h)^2}$$

$$= -2\frac{x}{x^4} = -2/x^3$$

b) Now find the equation of the tangent to the graph of the given function at $x = 2$.

The slope at $x = 2$ is $-2/2^3 = -1/4$.

The tangent has slope $-1/4$ and passes through $(2, 1/4)$.

Put $y = -\frac{1}{4}x + c$, substitute $(2, 1/4)$ for (x, y) and solve for c:

$1/4 = -1/4(2) + c \Rightarrow c = 3/4$. The tangent is $\boxed{y = -1/4 x + 3/4}$.

c) For what value(s) of x is the slope of the graph equal to $\frac{2}{27}$?

We need to solve $f'(x) = -2/x^3 = 2/27$.

$\Rightarrow x^3 = -27$

$\Rightarrow x = -3$.

The slope is $2/27$ at $x = -3$ (and nowhere else).

d) Assume now we have another function : $g(x) = \sqrt{4-x}$

i) What are the values of

$(g \circ f)(\frac{1}{2})$, $(g \circ f)(-\frac{1}{2})$, *and* $(g \circ f)(\frac{1}{\sqrt{3}})$?

$(g \circ f)(x) = g(f(x)) = \sqrt{4 - 1/x^2}$

$(g \circ f)(1/2) = \sqrt{4 - 1/(1/2)^2} = \sqrt{4-4} = 0$

$(g \circ f)(-1/2) = \sqrt{4 - 1/(-1/2)^2} = \sqrt{4-4} = 0$

$(g \circ f)(1/\sqrt{3}) = \sqrt{4 - 1/(1/\sqrt{3})^2} = \sqrt{4-3} = 1$

ii) What is the domain of the function $g \circ f$?

$g \circ f$ is defined wherever $4 - 1/x^2 \geq 0$.

$\Leftrightarrow \quad 4 \geq 1/x^2$

$\Leftrightarrow \quad 1/4 \leq x^2$

$\Leftrightarrow \quad 1/2 \leq |x|$

So the domain of $g \circ f$ is $(-\infty, -1/2] \cup [1/2, \infty)$.

4. (6 Points) *Assume that, for all values of* x *near 0, the unknown function* $g(x)$ *satisfies the condition:*

$$2 + \frac{\tan(2x)}{2x} < g(x) < 4 - \frac{\sin x}{x}$$

Apply the sandwich theorem to calculate: $\lim_{x \to 0}[g(x)]$.

The sandwich theorem says: if $f(x) \leq g(x) \leq h(x)$ for all x near c, and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$,

then $\lim_{x \to c} g(x) = L$.

We have $\lim_{x \to 0}\left(2 + \frac{\tan(2x)}{2x}\right) = 2 + \lim_{x \to 0}\left(\frac{\sin(2x)}{2x \cdot \cos(2x)}\right) = 2 + \lim_{x \to 0}\left(\frac{\sin(2x)}{2x}\right) \cdot \lim_{x \to 0}\left(\frac{1}{\cos(2x)}\right)$

$= 2 + 1 = 3$

and $\lim_{x \to 0}\left(4 - \frac{\sin x}{x}\right) = 4 - \lim_{x \to 0}\left(\frac{\sin x}{x}\right) = 4 - 1 = 3$.

So the sandwich theorem applies, and we deduce

$\lim_{x \to 0} g(x) = 3$.

5. (16 Points) Evaluate the following limits:

a) $\lim_{t\to\infty}\frac{3+2t-\cos t}{2\sin t-1+4t}$.

$$\lim_{t\to\infty}\frac{3+2t-\cos t}{2\sin t-1+4t} = \lim_{t\to\infty}\frac{3/t+2-\frac{\cos t}{t}}{2\frac{\sin t}{t}-1/t+4} = \frac{2}{4} = 1/2$$

Since $-1\le \sin t\le 1$, we have $-1/t \le \frac{\sin t}{t}\le 1/t$, so $\lim_{t\to\infty}\left(\frac{\sin t}{t}\right)=0$ by the Sandwich Theorem. Similarly $\lim_{t\to\infty}\frac{\cos t}{t}=0$.

b) $\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}$.

$$\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x\to 0}\frac{(\sqrt{x+2}-\sqrt{2})(\sqrt{x+2}+\sqrt{2})}{x(\sqrt{x+2}+\sqrt{2})} = \lim_{x\to 0}\frac{x+2-2}{x(\sqrt{x+2}+\sqrt{2})}$$

$$= \lim_{x\to 0}\frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}.$$

c) $\lim_{z\to 0}\frac{1-\cos^2(4z)}{5z^2}$.

$$\lim_{z\to 0}\frac{1-\cos^2(4z)}{5z^2} = \lim_{z\to 0}\frac{\sin^2(4z)}{5z^2} = \lim_{z\to 0}\frac{16}{5}\left(\frac{\sin(4z)}{4z}\right)^2 = \frac{16}{5}$$

d) $\lim_{x\to\infty}\frac{2x^{-1}-x^{-4}}{x^{-2}+5x^{-3}}$

$$\lim_{x\to\infty}\frac{2x^{-1}-x^{-4}}{x^{-2}+5x^{-3}} = \lim_{x\to\infty}\frac{2x+x^{-2}}{1+5x^{-1}} = \frac{\lim_{x\to\infty}(2x)+\lim_{x\to\infty}(x^{-2})}{1+\lim_{x\to\infty}(5x^{-1})} = \infty$$

6. (6 Points) A function $f(x)$ is assumed to be continuous over the closed interval [-4,4].The following table gives a selection of values of f

x	$f(x)$
-4	5
-2	-6
0	-8
2	3
4	-10

Explain how you can make use of the intermediate value theorem, in order to locate roots lying between -4 and 4, of the equation: $f(x)=0$.

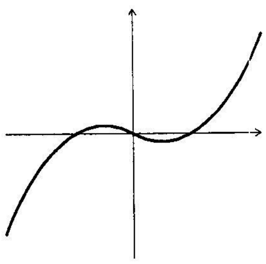
How many such roots are guaranteed by the mentioned theorem? Give adequate justification.

- $f(-4)=5$, $f(-2)=-6$. Since 0 is between 5 and -6, the Intermediate Value Theorem says there exists $c\in[-4,-2]$ such that $f(c)=0$.
- $f(0)=-8$, $f(2)=3$, so similarly there is $c\in[0,2]$ with $f(c)=0$.
- $f(2)=3$, $f(4)=-10$, so similarly there is $c\in[2,4]$ with $f(c)=0$.

All together, there must be at least 3 roots of $f(x)=0$ in $[-4,4]$.

7. (12 Points) Fill in the empty boxes to match the four graphs of the functions on the left to the graphs of their derivatives on the right.

Functions | **Derivatives**

$\boxed{A}$


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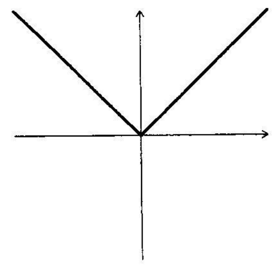

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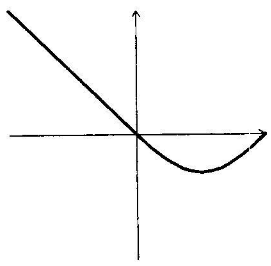

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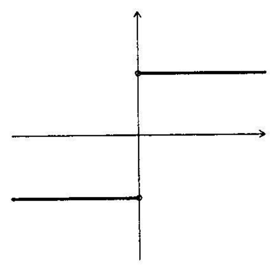

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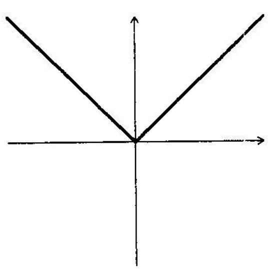

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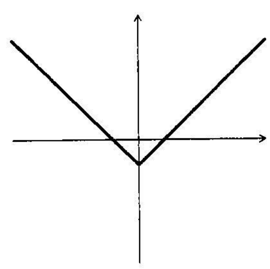

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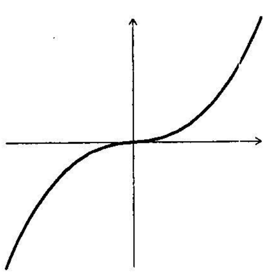

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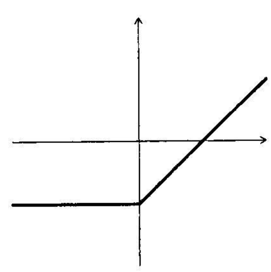


8. (8 Points)

a) i) Using the formula $\sin(A+B) = \sin A \cos B + \cos A \sin B$, show that:

$$\sin(\theta + \frac{\pi}{2}) = \cos\theta, \ \textit{for all } \theta.$$

$\sin(\theta + \pi/2) = \sin\theta \cos\pi/2 + \cos\theta \sin\pi/2$

$\boxed{\sin\pi/2 = 1, \ \cos\pi/2 = 0}$

$= 0 . \sin\theta + 1 . \cos\theta$

$= \cos\theta$ for all θ.

ii) What does this tell you about how the graphs of $y = \sin\theta$ *and* $y = \cos\theta$ are related?

The graph of $y = \cos\theta$ is the same as the graph of $y = \sin\theta$ shifted left by $\pi/2$.

b) Your friend mistakenly believes that $\sin(A+B) = \sin A + \sin B$ *for all A and B.*

i) Find a value of A and a value of B for which your friend's formula <u>does</u> work.

Take $A = B = 0$:

$\sin(0+0) = \sin(0) = 0$

$\sin(0) + \sin(0) = 0 + 0 = 0$ so the formula works.

ii) Show how you would convince your friend that his/her formula is wrong.

Find values of A and B for which it doesn't work.

For example, $A = B = \pi/2$

$\sin(A+B) = \sin(\pi) = 0$

but $\sin A + \sin B = \sin\pi/2 + \sin\pi/2 = 1 + 1 = 2$.

Answer the following by circling T (for true) or F (for false). You win 3 points for each correct match.

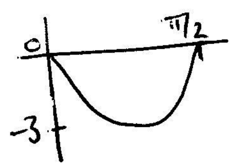
[T F] $\lim_{x\to 0}\cos(\frac{\pi - x}{4} + \sin(3x^{\frac{1}{3}})]) = 1$ The limit is $\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.

[T F] *If* $\tan x = \frac{2}{3}$ *and* $x \in [0, \frac{\pi}{2}]$, *then* $\sec x = \frac{\sqrt{13}}{3}$.*(Here x is measured in radians).*

$\sec^2 x = 1 + \tan^2 x = 1 + (\frac{2}{3})^2 = \frac{13}{9}$. In this interval cos x is positive, so sec x is too.

[T F] *The range of the function* $y = f(x) = -3\sin(2x)$; $0 \le x \le \frac{\pi}{2}$; *is* $[-3, 0]$.

Graph:



[T F] $\lfloor 6.1 + \lceil -3.5 \rceil \rfloor = 4$

$\lfloor 6.1 + \lceil -3.5 \rceil \rfloor = \lfloor 6.1 + (-3) \rfloor = \lfloor 3.1 \rfloor = 3$

[T F] $\lim_{x\to\infty} \frac{5 + x - 3x^3}{2x^3 - 4x^2 + 7} = -\frac{3}{2}$.

[T F] $\lim_{x\to 3^-} \frac{|x^2 - 9|}{x - 3} = -6$

If x is slightly less than 3, then $x^2 < 9$, so $|x^2 - 9| = 9 - x^2$.

Then $\frac{|x^2 - 9|}{x - 3} = \frac{(3-x)(3+x)}{x-3} = -(3+x)$, so the limit is -6

[T F] $y = f(x) = 3x^4 - 5$ *is an odd function.* Actually it is an even function.

[T F]

When the graph of the function $y = 1 + \frac{1}{x^2}$ *is stretched horizontally by a factor of* 5, *we get the graph of* $y = 1 + \frac{25}{x^2}$.

Substituting $\frac{x}{5}$ for x gives

$y = \frac{25}{x^2}$.